LIOUVILLE'S AND STACKEL'S THEOREMS FOR HYPERBOLIC VARIATIONAL PROBLEMS

(TEOREMY LIUVILLIA I SHTEKKELIA DLIA GIPERBOLICHESKIKH VARIATSIONNYKH ZADACH)

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In the work [1] it was shown that the Hamilton-Jacobi method of integrating canonical systems can be extended to the case of hyperbolic problems. Below there is considered a generalization of known conditions for the integrability of the Hamilton-Jacobi equation, namely, the theorems of Liouville and Stäckel.

1. The generalized theorem of Liouville. Suppose that the kinetic and potential energies of a system (of Liouville) have, respectively, the following forms

$$T = \frac{1}{2}\int_{x_0}^{x_1} A(z, x)\,dx \int_{x_0}^{x_1} B(z, x)\, z_y^2\,dx, \qquad \pi = \left[\int_{x_0}^{x_1} A(z, x)\,dx\right]^{-1}\int_{x_0}^{x_1} \Pi(z, x)\,dx \quad (1.1)$$

Then the Hamilton-Jacobi equation [1]

$$\frac{\partial S}{\partial y} + H\left(z;\ \frac{\delta S}{\delta z}\right) = 0 \qquad (1.2)$$

can be solved by quadratures.

Proof. For the Liouville system the canonical impulse q is given by

$$q = \frac{\delta(T - \pi)}{\delta z_y} = B(z, x)\, z_y \int_{x_0}^{x_1} A(z, x)\,dx \qquad (1.3)$$

Whence,

$$z_y = q\left[B(z, x)\int_{x_0}^{x_1} A(z, x)\,dx\right]^{-1} \qquad (1.4)$$

With the functional

$$H = \frac{1}{2}\left(\int_{x_0}^{x_1} A\,(z,\ x)\,dx\right)^{-1}\int_{x_0}^{x_1}\left[\frac{q^2}{B\,(z,\ x)} + 2\Pi\,(z,\ x)\right]dx$$

we construct the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial y} + \frac{1}{2}\left(\int_{x_0}^{x_1} A\,(z,\ x)\,dx\right)^{-1}\int_{x_0}^{x_1}\left[\frac{1}{B\,(z,\ x)}\left(\frac{\delta S}{\delta z}\right)^2 + 2\Pi\,(z,\ x)\right]dx = 0 \qquad (1.5)$$

Let us try to find S in the form

$$S = W - Ey \qquad (E = \text{const}) \qquad (1.6)$$

This is possible since H does not depend explicitly on y. For W we obtain the equation

$$\int_{x_0}^{x_1}\left[\frac{1}{B\,(z,\ x)}\left(\frac{\delta W}{\delta z}\right)^2 + 2\,(\Pi\,(z,\ x) - EA\,(z,\ x))\right]dx = 0 \qquad (1.7)$$

We obtain an equation in functional derivates*

$$\left(\frac{\delta W}{\delta z}\right)^2 = -2B\,(z,\ x)\,[\Pi\,(z,\ x) - EA\,(z,\ x) - \sigma\,(x)] \qquad (1.8)$$

with the additional condition

$$\int_{x_0}^{x_1} \sigma\,(x)\,dx = 0 \qquad (1.9)$$

The functional W is determined by means of (1.8) in the form of a functional integral

$$W = \int dx \int_{0}^{z(x)} \sqrt{2B\,(z,\ x)\,[EA\,(z,\ x) + \sigma\,(x) - \Pi\,(z,\ x)]}\,dz\,(x) \qquad (1.10)$$

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* The circumstance that the equation (1.7) contains in addition to $\delta W/\delta z$ also the functions z and x, makes it possible to seek W in the form

$$\int_{x_0}^{x_1} w\,(z,\ x)\,dx$$

where w is a function of its arguments. Hereby, $\delta W/\delta z = \partial w/\partial z$; the vanishing of the integral has to be identical relative to z. From this we conclude that the integrand is independent of z; it is a function of x only. This implies (1.8).

It is obvious that the condition of symmetry is satisfied

$$\frac{\delta}{\delta z(\xi)} \sqrt{2B(z,\eta)[EA(z,\eta)+\sigma(\eta)-\Pi(z,\eta)]} = \\ = \frac{\delta}{\delta z(\eta)} \sqrt{2B(z,\xi)[EA(z,\xi)+\sigma(\xi)-\Pi(z,\xi)]} \tag{1.11}$$

The functional integral (1.10), therefore, reduces to an ordinary one [2], and we obtain*

$$W = \int dx \int_0^1 \sqrt{2B(zt,x)[EA(zt,x)+\sigma(x)-\Pi(zt,x)]}\, z dt \tag{1.12}$$

Taking the variation of the functional $S = W - Ey$ with respect to σ and z, we obtain, in accordance with the theorem of Jacobi [1], the finite equations of motion.

2. The generalized theorem of Stäckel. Just as in the classical case, Liouville's theorem is a particular case of a more general result; namely, a generalized Stäckel theorem is true. Let the kinetic energy and the potential energy of a system (Stäckel) be given:

$$T = \frac{1}{2}\int_{x_0}^{x_1} \frac{z_y^2\, dx}{\psi[z,x]}, \qquad \pi = \int_{x_0}^{x_1} \psi[z,x]\,\Pi(z,x)\,dx$$

where $\psi[z, x]$ is a functional. We introduce the function $\varphi(z, x; \zeta)$ by the relations

$$\int_{x_0}^{x_1} \varphi(z,x;\zeta)\psi[z,x]\,dx = \begin{cases} 0 & (x_0 \leqslant \zeta < x_1) \\ 1 & (\zeta = x_1) \end{cases} \tag{2.1}$$

Then the Hamilton-Jacobi equation can be solved by quadratures. Writing out the complete integral of Levy in the form (1.6), we arrive, as above, at an equation of the type (1.7)

$$\int_{x_0}^{x_1} \psi[z;x]\left\{\frac{1}{2}q^2[z;x] + \Pi(z,x)\right\} dx = E$$

Keeping in mind condition (2.1), we find

$$\frac{1}{2}\left(\frac{\delta W}{\delta z}\right)^2 = -\Pi(z,x) + \int_{x_0}^{x_1} \rho(\zeta)\,\varphi(z,x;\zeta)\,d\zeta + E\varphi(z,x;x_1)$$

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* The result could have been foreseen quite easily (see the reference in the preceding section).

where $\rho(\zeta)$ is a parametric function [1]. The right-hand side of the last equation is a function (not a functional) of z. Just as in the proof of Liouville's theorem, we find that the functional integral

$$W = \int dx \int_0^{z(x)} \left(2\left[E\varphi(z, x; x_1) + \int_{x_0}^{x_1} \rho(\zeta)\,\varphi(z, x; \zeta)\,d\zeta - \Pi(z, x)\right]\right)^{1/2} dz(x)$$

reduces to an ordinary one.

For the sake of completeness of the proof of this theorem (as well as of the preceding one) one should convince himself that the obtained solutions of the Hamilton-Jacobi equation are the complete integrals of Levy. For this purpose it is simpler to return to the finite-dimensional analog of the performed steps, where the made assertion is obviously true. The transition to the limit to "an infinite number of degrees of freedoms" will then complete the proof.

One can prove, just as in the classical case [3], that the conditions of the Stäckel theorem are necessary for the representability of the complete integral of Levy in the form

$$\int dx \int_0^{z(x)} F(z, x)\,dz(x)$$

The application of Liouville's as well as Stäckel theorems directly in the given formulations is limited to problems in which the potential energy of the system does not depend on the derivative with respect to x of the sought function. As a rule, this requirement is not fulfilled, and one should, if this is possible, introduce such general coordinates for which this requirement is fulfilled. For an infinite string, for example, such a coordinate is provided by the Fourier transform of the resulting function $z(x, y)$; the canonical conjugate impulse is determined in a corresponding way.

BIBLIOGRAPHY

1. Lur'e, K.A., O metode Gamiltona-Iakobi v variatsionnykh zadachakh s chastnymi proizvodnymi (On the Hamilton-Jacobi method in variational problems with partial derivatives). *PMM* Vol. 27, No. 2, 1963.

2. Svidzinskii, A.V., Krivolineinyi integral v funktsional'nom prostranstve (Curvilinear integral in a function space). *Izv. vyssh. uchebn. zaved., matematika*, No. 1(8), 1959.

3. Lur'e, A.I., *Analiticheskaia mekhanika (Analytical mechanics)*. Fizmatgiz, 1961.

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